Sine Limit Examples

We begin by introducing the following two rules:

$(i) \quad \displaystyle\lim_{x \to 0} \frac{\sin kx}{kx} = 1 \qquad \text{and} \qquad (ii) \quad \displaystyle\lim_{x \to 0} \frac{kx}{\sin kx} = 1 \qquad \text{for } k \in \mathbb{R} \text{ and } k \neq 0.$

For example we can say:

$$\lim_{x \to 0} \frac{\sin 5x}{5x} = 1 \qquad \text{or} \qquad \lim_{\theta \to 0} \frac{3\theta}{\sin 3\theta} = 1 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin\left(-\frac{1}{2}x\right)}{-\frac{1}{2}x} = 1$$

Example 1

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin 4x}{x} &= \lim_{x \to 0} \frac{4\sin 4x}{4x} && \textit{Multiplying top and bottom by 4} \\
&= 4 \lim_{x \to 0} \frac{\sin 4x}{4x} \\
&= 4(1) && \textit{Applying rule (i) above} \\
&= 4.
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\lim_{x \to 0} \frac{3x}{\sin 6x} &= \lim_{x \to 0} \frac{6x}{2\sin 6x} && \textit{Multiplying top and bottom by 2} \\
&= \frac{1}{2} \lim_{x \to 0} \frac{6x}{\sin 6x} \\
&= \frac{1}{2}(1) && \textit{Applying rule (ii) above} \\
&= \frac{1}{2}.
\end{aligned}
$$

Example 3

$$
\begin{aligned}
\lim_{x \to 0} \frac{3x}{\tan 4x} &= \lim_{x \to 0} \frac{3x}{\frac{\sin 4x}{\cos 4x}} \\
&= \lim_{x \to 0} \frac{3x \cos 4x}{\sin 4x} \\
&= 3 \lim_{x \to 0} \left(\cos 4x\right)\left(\frac{x}{\sin 4x}\right) \\
&= 3 \left(\lim_{x \to 0} \cos 4x\right)\left(\lim_{x \to 0} \frac{x}{\sin 4x}\right) \\
&= 3 \left(\lim_{x \to 0} \cos 4x\right)\left(\frac{1}{4} \lim_{x \to 0} \frac{4x}{\sin 4x}\right) \\
&= \frac{3}{4}.
\end{aligned}
$$

Try the following exercises for practice:

(a)
$$\lim_{x \to 0} \frac{\sin 7x}{x}$$

(b)
$$\lim_{x \to 0} \frac{x}{\sin 8x}$$

(c)
$$\lim_{x \to 0} \frac{\sin 7x}{3x}$$

(d)
$$\lim_{t \to 0} \frac{\tan t}{t}$$

(e)
$$\lim_{\theta \to 0} \frac{5 \sin \theta}{\theta}$$

Solutions
(a) 7
(b) $\frac{1}{8}$
(c) $\frac{7}{3}$
(d) 1
(e) 5