Limits to Infinity

In this handout we will examine limits of the form

$$\lim_{x\to\infty}\frac{g(x)}{h(x)} \qquad \text{or} \qquad \lim_{x\to-\infty}\frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials.
The important thing to remark is that we are taking the limit to ∞ or $-\infty$ of a rational function. We note that

$$\lim_{x\to\infty}\frac{\text{Any finite number}}{x^n}=0 \qquad \text{or} \qquad \lim_{x\to-\infty}\frac{\text{Any finite number}}{x^n}=0 \quad \text{for } n\in\mathbb{N}.$$

Using the above facts, we try to rewrite our functions using terms like $\frac{c}{x^n}$ where $c\in\mathbb{R}$. To do this we divide every term in the numerator and denominator by the highest power or x that appears in the function.

Example 1

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2-9x+8}{5x^2-x} &= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{9x}{x^2}+\frac{8}{x^2}}{\frac{5x^2}{x^2}-\frac{x}{x^2}}\\ &= \lim_{x\to\infty}\frac{1-\frac{9}{x}+\frac{8}{x^2}}{5-\frac{1}{x}}.\end{aligned}$$

We are now in a position to take the limit. We get

$$\begin{aligned}\lim_{x\to\infty}\frac{1-\frac{9}{x}+\frac{8}{x^2}}{5-\frac{1}{x}} &= \frac{1-0+0}{5-0}\\ &= \frac{1}{5}.\end{aligned}$$

In summary, we have found that

$$\lim_{x\to\infty}\frac{x^2-9x+8}{5x^2-x}=\frac{1}{5}.$$

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Example 2

$$\begin{aligned}
\lim_{x\to-\infty}\frac{3x^2-6x+2}{5x^3-2x^2+6x+7} &= \lim_{x\to-\infty}\frac{\frac{3x^2}{x^3}-\frac{6x}{x^3}+\frac{2}{x^3}}{\frac{5x^3}{x^3}-\frac{2x^2}{x^3}+\frac{6x}{x^3}+\frac{7}{x^3}} \\
&= \lim_{x\to-\infty}\frac{\frac{3}{x}-\frac{6}{x^2}+\frac{2}{x^3}}{5-\frac{2}{x}+\frac{6}{x^2}+\frac{7}{x^3}} \\
&= \frac{0-0+0}{5-0+0+0} \\
&= 0.
\end{aligned}$$

Example 3

$$\begin{aligned}
\lim_{x\to\infty}\frac{4x^3+5}{5x^2-1} &= \lim_{x\to\infty}\frac{\frac{4x^3}{x^3}+\frac{5}{x^3}}{\frac{5x^2}{x^3}-\frac{1}{x^3}} \\
&= \lim_{x\to\infty}\frac{4+\frac{5}{x^3}}{\frac{5}{x}-\frac{1}{x^3}} \\
&= \infty.
\end{aligned}$$

Try the following exercises for practice.

Calculate the following limits:

(a)

$$\lim_{x\to\infty}\frac{x^4-2x^2-x}{2x^4+3x^3+4}$$

(b)

$$\lim_{x\to\infty}\frac{3x^3-2x^2-2}{x^5-x^2+1}$$

(c)

$$\lim_{x\to\infty}\frac{3x^5+6x^2}{7x^4-3x^2-4}$$

(d)

$$\lim_{x\to-\infty}\frac{3x^6+4x^5-x^3-2x^2-1}{5x^6-4x^3+3x+2}$$

(e)

$$\lim_{x\to-\infty}\frac{5x^2-1}{3x^6-2x^2-7}$$

Solutions

(a) $\frac{1}{2}$ (b) 0 (c) ∞ (d) $\frac{3}{5}$ (e) 0