Using a table of derivatives

mc-TY-table1-2009-1

In this unit we construct a Table of Derivatives of commonly occurring functions. This is done using the knowledge gained in previous units on differentiation from first principles.

Rules, known as linearity rules, for constant multiples of functions, and for the sum/difference of two functions are also given and illustrated with examples.

Finally, the table is extended further by making use of the chain rule for differentiating a function of a function.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- construct and use a table of derivatives of commonly occurring functions

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1. Introduction

Other units have explained how to differentiate all the common functions, such as x^n, $\cos x$, $\sin x$, e^x, $\ln x$ and so on. Usually this has been done from first principles. In this unit we pull all these results together and construct a table of standard derivatives which you can consult as the need arises.

2. A Table of Derivatives

Commonly occurring functions and their derivatives are given in the Table below.

function $f(x)$	derivative $\dfrac{df}{dx}$ or $f'(x)$	
c	0	c is any constant
x	1	
$2x$	2	
x^n	nx^{n-1}	n is any real number
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\ln x$	$\dfrac{1}{x}$	

We can make this table more useful by extending the range of functions it includes. We can do this using rules, known as **linearity rules**: these are the **constant multiplier rule** and the **addition rule**.

3. The constant multiplier rule

Suppose we have a function $f(x)$ and multiply it by a constant, c say. The derivative of this is given by the following rule:

$$\frac{d}{dx}\left(c\,f(x)\right) = c\,\frac{df}{dx}$$

In other words, we simply differentiate the function and multiply the result by the constant c.

Examples

If $y = 2\sin x$ then $\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\sin x) = 2\cos x$.

If $y = -5\sin x$ then $\dfrac{dy}{dx} = -5\dfrac{d}{dx}(\sin x) = -5\cos x$.

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Proof from first principles of the constant multiplier rule
Consider the function $g(x) = cf(x)$ where c is a constant.
Using the definition of the derivative of $g(x)$ we have

$$\begin{aligned} g'(x) &= \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{c\,f(x+\delta x) - c\,f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{c(f(x+\delta x) - f(x))}{\delta x} \end{aligned}$$

The constant c is unaffected by the limiting process and so can be taken outside the limit:

$$g'(x) = c \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

But the term following the c on the right hand side is just the definition of the derivative of $f(x)$. So we have

$$g'(x) = c\,\frac{df}{dx} = c\,f'(x)$$

Key Point

The constant multiplier rule:

$$\frac{d}{dx}\,(c\,f(x)) = c\,\frac{df}{dx}$$

Exercise 1

Find the derivative of each of the following:

a)	$5x^4$	b)	$12x$	c)	$4x^{-2}$	d)	$8\cos x$	e)	$-3\cos x$
f)	$2\mathrm{e}^x$	g)	$3\ln x$	h)	$-7\mathrm{e}^x$	i)	$-2\sin x$	j)	$-4\ln x$

4. The addition and subtraction rules

The first of these rules enables us to differentiate the sum of two functions, e.g. $f(x) + g(x)$. The rule states that to differentiate this sum we simply differentiate each term separately and then add the results:

$$\frac{d}{dx}\,(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

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Similarly, if we have the difference of two functions:

$$\frac{d}{dx}\left(f(x) - g(x)\right) = \frac{df}{dx} - \frac{dg}{dx}$$

Key Point

Sum and difference rules:

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$$

These rules can be added to the Table given on Page 2.

Example

Suppose we wish to differentiate $y = 2x^3 - 6\cos x$.

We differentiate each term separately, and make use of the constant multiplier rule:

$$\begin{aligned}
\frac{d}{dx}\left(2x^3 - 6\cos x\right) &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(6\cos x) \\
&= 2\frac{d}{dx}(x^3) - 6\frac{d}{dx}(\cos x) \\
&= 2(3x^2) - 6(-\sin x) \\
&= 6x^2 + 6\sin x
\end{aligned}$$

5. Further extensions to the Table

In this section we extend the Table by looking at functions of the form $\sin mx$, $\cos mx$, e^{mx} and $\ln mx$.

Example

Suppose we wish to differentiate $y = \sin mx$ in order to find $\frac{dy}{dx}$.

We begin by making the substitution $u = mx$. This simplifies the original function to give $y = \sin u$.

To find $\frac{dy}{dx}$ we use a rule (called the chain rule, or the function of a function rule) which states

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This rule is dealt with at length in another unit. For now we will simply quote and use it.

We need to calculate $\frac{dy}{du}$: since $y = \sin u$ it follows that $\frac{dy}{du} = \cos u$.

We also need to calculate $\frac{du}{dx}$: since $u = mx$ it follows that $\frac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \cos u\times m\\ &= m\cos u\\ &= m\cos mx\end{aligned}$$

since $u = mx$.

We have shown that if $y = \sin mx$ then $\dfrac{dy}{dx} = m\cos mx$.

In a similar way it is straightforward to show that if $y = \cos mx$ then $\dfrac{dy}{dx} = -m\sin mx$.

Example

Suppose we wish to differentiate $y = \mathrm{e}^{mx}$.

Again, we substitute $u = mx$ so that $y = \mathrm{e}^{u}$. We then use the chain rule:

We need to calculate $\dfrac{dy}{du}$: since $y = \mathrm{e}^{u}$ it follows that $\dfrac{dy}{du} = \mathrm{e}^{u}$.

We also need to calculate $\dfrac{du}{dx}$: since $u = mx$ it follows that $\dfrac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \mathrm{e}^{u}\times m\\ &= m\mathrm{e}^{u}\\ &= m\mathrm{e}^{mx}\end{aligned}$$

since $u = mx$.

We have shown that if $y = \mathrm{e}^{mx}$ then $\dfrac{dy}{dx} = m\mathrm{e}^{mx}$.

Example

Suppose we wish to differentiate $y = \ln mx$.

Again, we substitute $u = mx$ so that $y = \ln u$. We then use the chain rule:

We need to calculate $\dfrac{dy}{du}$: since $y = \ln u$ it follows that $\dfrac{dy}{du} = \dfrac{1}{u}$.

We also need to calculate $\dfrac{du}{dx}$: since $u = mx$ it follows that $\dfrac{du}{dx} = m$ because m is a constant.

Substituting into the chain rule we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= \frac{1}{u}\times m\\ &= m\frac{1}{mx}\\ &= \frac{1}{x}\end{aligned}$$

We have shown that if $y = \ln mx$ for constant m, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

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This result could also have been obtained by using the laws of logarithms to rewrite $y = \ln mx$ as $y = \ln m + \ln x$. Then we could differentiate this sum, term by term. The first term has derivative zero. This is because the the logarithm of a constant is still a constant and so its derivative is zero. The derivative of the second term is simply $\frac{1}{x}$.

Example

Suppose we wish to differentiate $y = \ln(ax + b)$ where a and b are constants.

This time we substitute $u = ax + b$ so that $y = \ln u$. We then use the chain rule:
We need to calculate $\dfrac{dy}{du}$: since $y = \ln u$ it follows that $\dfrac{dy}{du} = \dfrac{1}{u}$.
We also need to calculate $\dfrac{du}{dx}$: since $u = ax + b$ it follows that $\dfrac{du}{dx} = a$ because a and b are constants.

Substituting into the chain rule we find

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times a \\ &= \frac{a}{u} \\ &= \frac{a}{ax+b} \end{aligned}$$

since $u = ax + b$.
We have shown that if $y = \ln(ax + b)$ for constants a and b, then $\dfrac{dy}{dx} = \dfrac{a}{ax+b}$.

The results we have generated in the preceding sections can be added to the table of derivatives given on page 2 to produce a more complete and thereby more useful table:

function $f(x)$	derivative $\dfrac{df}{dx}$ or $f'(x)$	
c	0	c is any constant
x	1	
$2x$	2	
x^n	nx^{n-1}	n is any real number
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\ln x$	$\dfrac{1}{x}$	
$cf(x)$	$cf'(x)$	c constant
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$	
$\sin mx$	$m \cos mx$	m is a constant
$\cos mx$	$-m \sin mx$	m is a constant
e^{mx}	$m\mathrm{e}^{mx}$	m is a constant
$\ln mx$	$\frac{1}{x}$	
$\ln(ax+b)$	$\frac{a}{ax+b}$	

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Exercise 2
Find the derivative of each of the following:

a)	$\sin 4x$	b)	e^{5x}	c)	$\cos 3x$	d)	$\ln 5x$
e)	$2\sin 3x$	f)	$4e^{-2x}$	g)	$4\cos(-2x)$	h)	$\ln(3x+2)$
i)	$4\sin 2x + 3\cos 3x$	j)	$e^{2x} - e^{-2x}$	k)	$\ln 2x + 2\sin 3x$	l)	$4\cos x - 2\ln(x+4)$

Answers

Exercise 1

a) $20x^3$ b) 12 c) $-8x^{-3}$ d) $-8\sin x$ e) $3\sin x$
f) $2e^x$ g) $\dfrac{3}{x}$ h) $-7e^x$ i) $-2\cos x$ j) $-\dfrac{4}{x}$

Exercise 2

a) $4\cos 4x$ b) $5e^{5x}$ c) $-3\sin 3x$ d) $\dfrac{1}{x}$ e) $6\cos 3x$ f) $-8e^{-2x}$ g) $8\sin(-2x)$
h) $\dfrac{3}{3x+2}$ i) $8\cos 2x - 9\sin 3x$ j) $2e^{2x} + 2e^{-2x}$ k) $\dfrac{1}{x} + 6\cos 3x$ l) $-4\sin x - \dfrac{2}{x+4}$

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Extending the table of derivatives

mc-TY-table2-2009-1

In this unit we continue to build up The Table of Derivatives using rules described in other units.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate trigonometric functions
- differentiate inverse trigonometric functions
- differentiate a^x.

Contents

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1. Introduction

In this unit we construct several new entries for the Table of Derivatives using standard rules and results obtained previously. The table we will construct is shown here:

function $f(x)$	derivative $\dfrac{df}{dx}$ or $f'(x)$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\text{cosec}^2 x$
$\text{cosec } x = \frac{1}{\sin x}$	$-\text{cosec } x \cot x$
$\tan mx$	$m \sec^2 mx$
$\sec mx$	$m \sec mx \tan mx$
$\cot mx$	$-m\, \text{cosec}^2 mx$
$\text{cosec } mx$	$-m\, \text{cosec } mx \cot mx$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
a^x	$a^x \ln a$

Most of these new entries are derived in the following examples.

2. Examples

Example 1

Suppose we wish to differentiate $y = \tan x$. Note that we can write this as $y = \tan x = \frac{\sin x}{\cos x}$. Because this is a quotient we can use the quotient rule to perform the differentiation.

The quotient rule states:

$$\text{if} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So to differentiate $y = \frac{\sin x}{\cos x}$ we take $u = \sin x$ and $v = \cos x$. Then

$$\frac{du}{dx} = \cos x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x$$

Then, applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

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A key trigonometric identity states that $\cos^2 x + \sin^2 x = 1$ and so this simplifies to

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

which can also be written as $\sec^2 x$. So the derivative of $\tan x$ is $\sec^2 x$.

Example 2

Suppose we wish to differentiate $y = \sec x$.

Note that we can write this as $y = \sec x = \frac{1}{\cos x}$. Because this is a quotient we can again use the quotient rule to perform the differentiation.

The quotient rule states:

$$\text{if} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So to differentiate $y = \frac{1}{\cos x}$ we take $u = 1$ and $v = \cos x$. Then

$$\frac{du}{dx} = 0 \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x$$

Then, applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x(0) - 1(-\sin x)}{(\cos x)^2} \\ &= \frac{\sin x}{\cos^2 x} \end{aligned}$$

this can be written as

$$\frac{dy}{dx} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$$

So the derivative of $\sec x$ is $\sec x \tan x$.

Example 3

Suppose we wish to differentiate $y = \tan mx$ where m is a constant.

We make a substitution to simplify this function. Suppose we let $u = mx$ so that $y = \tan u$. We can use the chain rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In this case, since $u = mx$ then $\frac{du}{dx} = m$.
Since $y = \tan u$, we have just seen that $\frac{dy}{du} = \sec^2 u$,
So, the chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 u \times m \\ &= m \sec^2 mx \end{aligned}$$

Example 4

Suppose we wish to differentiate $y = \operatorname{cosec} mx$ where m is a constant.

We make a substitution to simplify this function. Suppose we let $u = mx$ so that $y = \operatorname{cosec} u$. We can again use the chain rule to find $\frac{dy}{dx}$: In this case, since $u = mx$ then $\frac{du}{dx} = m$.
Since $y = \operatorname{cosec} u$, we note from the Table on page 2 that $\frac{dy}{du} = -\operatorname{cosec} u \cot u$,
So, the chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= -\operatorname{cosec} u \cot u \times m \\ &= -m \operatorname{cosec} mx \cot mx \end{aligned}$$

Example 5

Suppose we wish to differentiate $y = \sin^{-1} x$.
We proceed by rewriting this as $\sin y = x$. We then differentiate both sides with respect to x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

The right hand side is straightforward because the derivative of x with respect to x is just 1. The left hand side needs more care because we need to differentiate a function of y, that is $\sin y$, with respect to x. We do this implicitly as follows:

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dy}(\sin y) \times \frac{dy}{dx} \\ &= \cos y \frac{dy}{dx} \end{aligned}$$

(If necessary you should refer to the unit on implicit differentiation in order to understand this process.) Putting these results together we find

$$\cos y \frac{dy}{dx} = 1$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We now try to write the right hand side in terms of x. We can do this using the identity

$$\cos^2 y + \sin^2 y = 1$$

so that

$$\cos y = \sqrt{1 - \sin^2 y}$$

We take only the positive square root. This is because studying the graph of $y = \sin^{-1} x$ shows that its gradient, and hence $\frac{dy}{dx}$, is positive.
So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

Finally, recall that $\sin y = x$ so that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

So the derivative of $y = \sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$.
A similar argument can be used to find the derivative of $y = \cos^{-1} x$ and this is left as an exercise.

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Example 6

Suppose $y = \tan^{-1} x$ and we wish to find $\frac{dy}{dx}$. We proceed by rewriting this as $\tan y = x$. We then differentiate both sides with respect to x:

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

The right hand side is straightforward because the derivative of x with respect to x is just 1. The left hand side needs more care because we need to differentiate a function of y, that is $\tan y$, with respect to x. We do this implicitly as follows:

$$\begin{aligned}\frac{d}{dx}(\tan y) &= \frac{d}{dy}(\tan y) \times \frac{dy}{dx} \\ &= \sec^2 y \frac{dy}{dx}\end{aligned}$$

Putting these results together we find

$$\sec^2 y \frac{dy}{dx} = 1$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}\end{aligned}$$

Here we have made use of the trigonometric identity $1 + \tan^2 y = \sec^2 y$. So the derivative of $y = \tan^{-1} x$ is $\frac{1}{1+x^2}$.

Example 7

Suppose we want to differentiate $y = a^x$. We proceed by taking logarithms of both sides:

$$\ln y = \ln a^x = x \ln a$$

using the laws of logarithms.
Differentiating both sides with respect to x gives

$$\frac{d}{dx} \ln y = \ln a$$

since $\ln a$ is a constant. Now, using the chain rule,

$$\frac{d}{dx} \ln y = \frac{d}{dy} \ln y \times \frac{dy}{dx}$$

so

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

Then

$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

Exercises

1. Use the extended table of derivatives in Section 1 to find the derivative of each of the following:

a)	$\tan 3x$	b)	$\cos^{-1} x$	c)	5^x	d)	$\cot 5x$
e)	$\cot 3x$	f)	2^x	g)	$\text{cosec}4x$	h)	$\tan^{-1} x$
i)	$\sec 3x$	j)	$\tan 4x$	k)	1^x	l)	$\sec\left(\frac{1}{2}x\right)$

2. By writing $\cot x = \dfrac{\cos x}{\sin x}$ and using the quotient rule find the derivative of $\cot x$.

3. By writing $\text{cosec}x = \dfrac{1}{\sin x}$ and using the quotient rule find the derivative of $\text{cosec}x$.

4. By implicitly differentiating $\cos y = x$ determine the derivative of $\cos^{-1} x$.

5. Use the chain rule to find the derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$.

Answers

1. a) $3\sec^2 3x$ b) $-\dfrac{1}{\sqrt{1-x^2}}$ c) $5^x \ln 5$ d) $-5\text{cosec}^2 5x$ e) $-3\text{cosec}^2 3x$
f) $2^x \ln 2$ g) $-4\text{cosec}4x \cot 4x$ h) $\dfrac{1}{1+x^2}$ i) $3\sec 3x \tan 3x$ j) $4\sec^2 4x$
k) 0 l) $\dfrac{1}{2}\sec\left(\dfrac{1}{2}x\right)\tan\left(\dfrac{1}{2}x\right)$

2. $-\text{cosec}^2 x$

3. $-\text{cosec}x \cot x$

4. $-\dfrac{1}{\sqrt{1-x^2}}$

5. $\dfrac{a}{a^2+x^2}$

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