Simultaneous linear equations

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The purpose of this section is to look at the solution of simultaneous linear equations. We will see that solving a pair of simultaneous equations is equivalent to finding the location of the point of intersection of two straight lines.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve pairs of simultaneous linear equations
- recognise that this is equivalent to finding the point of intersection of two straight line graphs

Contents

1. Introduction

The purpose of this section is to look at the solution of elementary simultaneous linear equations.

Before we do that, let's just have a look at a relatively straightforward single equation. The equation we are going to look at is

$$2x - y = 3$$

This is a linear equation. It is a linear equation because there are no terms involving x^2, y^2 or $x \times y$, or indeed any higher powers of x and y. The only terms we have got are terms in x, terms in y and some numbers. So this is a **linear equation**.

We can rearrange it so that we obtain y on its own on the left hand side. We can add y to each side so that we get

$$2x = 3 + y$$

Now let's take 3 away from each side.

$$2x - 3 = y$$

This gives us an expression for y: namely $y = 2x - 3$.

Suppose we choose a value for x, say $x = 1$, then y will be equal to:

$$y = 2 \times 1 - 3 = -1$$

Suppose we choose a different value for x, say $x = 2$.

$$y = 2 \times 2 - 3 = 1$$

Supppose we choose another value for x, say $x = 0$.

$$y = 2 \times 0 - 3 = -3$$

For every value of x we can generate a value of y.

We can plot these as points on a graph. We can plot the first as the point $(1, -1)$. We can plot the second one as the point $(2, 1)$, and the third one as the point $(0, -3)$ and so on. Plotting the points on a graph, as shown in Figure 1, we see that these three points lie on a straight line. This is the line with equation $y = 2x - 3$. It is a straight line and this is another reason for calling the equation a **linear equation**.

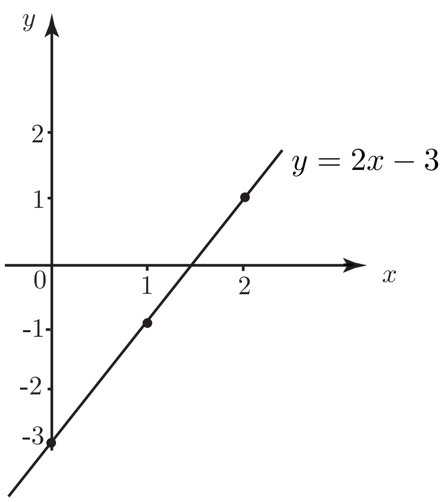

Figure 1. Graph of $y = 2x - 3$.

Suppose we take a second linear equation $3x + 2y = 8$ and plot its graph on the same figure. A quick way to achieve this is as follows.
When $x = 0$, $2y = 8$, so $y = 4$. Therefore the point $(0, 4)$ lies on the line.

When $y = 0$, $3x = 8$, so $x = \frac{8}{3} = 2\frac{2}{3}$. Therefore the point $(2\frac{2}{3}, 0)$ lies on the line.

Because this is a linear equation we know its graph is a straight line, so we can obtain this by joining up the points. Both straightline graphs are shown in Figure 2.

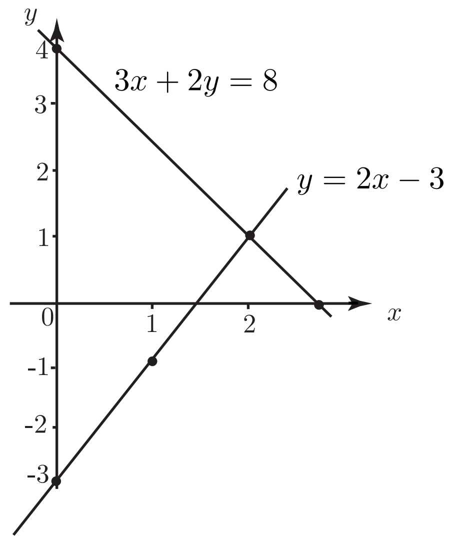

Figure 2. Graphs of $y = 2x - 3$ and $3x + 2y = 8$

When we solve a pair of simultaneous equations what we are actually looking for is the intersection of two straight lines because it is this point that satisfies both equations at the same time. From Figure 2 we see that this occurs at the point where $x = 2$ and $y = 1$.

Of course it could happen that we have two parallel lines; they would never meet, and hence the simultaneous equations would not have a solution. We shall observe this behaviour in one of the examples which follows.

Key Point

When solving a pair of simultaneous linear equations we are, in fact, finding a common point - the point of intersection of the two lines.

2. Solving simultaneous equations - method of substitution

How can we handle the two equations algebraically so that we do not have to draw graphs? We are going to look at two methods of solution. In this Section we will look at the first method - the method of substitution.

Let us return to the two equations we met in Section 1.

$$2x - y = 3 \quad (1)$$

$$3x + 2y = 8 \quad (2)$$

By rearranging Equation (1) we find

$$y = 2x - 3 \quad (3)$$

We can now substitute this expression for y into Equation (2).

$$\begin{aligned} 3x + 2(2x - 3) &= 8 \\ 3x + 4x - 6 &= 8 \\ 7x - 6 &= 8 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Finally, using Equation (3), $y = 2 \times 2 - 3 = 1$. So $x = 2$, $y = 1$ is the solution to the pair of simultaneous equations.

This solution should always be checked by substituting back into both original equations to ensure that the left- and right- hand sides are equal for these values of x and y. So, with $x = 2$, $y = 1$, the left-hand side of Equation (1) is $2(2) - 1 = 3$, which is the same as the right-hand side. With $x = 2$, $y = 1$, the left-hand side of Equation (2) is $3(2) + 2(1) = 8$, which is the same as the right-hand side.

Example

Let's have a look at another example using this particular method.

The example we are going to use is

$$7x + 2y = 47 \quad (1)$$

$$5x - 4y = 1 \quad (2)$$

Now we need to make a choice. We need to choose one of these two equations and re-arrange it to obtain an expression for y, or if we wish, for x

The choice is entirely ours and we have to make the choice based upon what we feel will be the simplest. Looking at a pair of equations like this, it is often difficult to know which is the simplest.

Let's choose Equation (2) and rearrange it to find an expression for x.

$$\begin{aligned} 5x - 4y &= 1 \\ 5x &= 1 + 4y \qquad \text{by adding } 4y \text{ to each side} \\ x &= \frac{1+4y}{5} \qquad \text{by dividing both sides by 5} \end{aligned}$$

We now use this expression for x and substitute it in Equation (1).

$$7\left(\frac{1+4y}{5}\right) + 2y = 47$$

Now multiply throughout by 5. Why? Because we want to get rid of the fraction and the way to do that is to multiply everything by 5.

$$7(1+4y) + 10y = 235$$

Now we need to multiply out the brackets

$$7 + 28y + 10y = 235$$

Gather the y's and subtract 7 from each side to get

$$38y = 228$$

So

$$y = \frac{228}{38} = 6$$

So we have established that $y = 6$. Having done this we can substitute it back into the equation that we first had for x.

$$x = \frac{1+4y}{5} = \frac{1+24}{5}$$

and so

$$x = 5$$

So again, we have our pair of values - our solution to the pair of simultaneous equations. In order to check that our solution is correct these values should be substituted into both equations to ensure they balance. So, with $x = 5$, $y = 6$, the left-hand side of Equation (1) is $7(5)+2(6) = 47$, which is the same as the right-hand side. With $x = 5$, $y = 6$, the left-hand side of Equation (2) is $5(5) - 4(6) = 1$, which is the same as the right-hand side.

Exercises

1. Solve the following pairs of simultaneous equations:

a) $\begin{aligned} y &= 2x + 3 \\ y &= 5x - 3 \end{aligned}$ b) $\begin{aligned} y &= 3x - 1 \\ 2x + 4y &= 10 \end{aligned}$ c) $\begin{aligned} 6x + y &= 4 \\ 5x + 2y &= 1 \end{aligned}$

d) $\begin{aligned} x - 3y &= 1 \\ 2x + 5y &= 35 \end{aligned}$ e) $\begin{aligned} 2x + \tfrac{1}{3}y &= 1 \\ 3x + 5y &= 6 \end{aligned}$ f) $\begin{aligned} 4x + 3y &= 5 \\ 2x - \tfrac{3}{4}y &= 1 \end{aligned}$

3. Solving simultaneous equations - method of elimination

We illustrate the second method by solving the simultaneous linear equations:

$$7x + 2y = 47 \quad (1)$$

$$5x - 4y = 1 \quad (2)$$

We are going to multiply Equation (1) by 2 because this will make the magnitude of the coefficients of y the same in both equations. Equation (1) becomes

$$14x + 4y = 94 \quad (3)$$

If we now add Equation (2) and Equation (3) we will find that the terms involving y disappear:

$$\begin{array}{rrrrcr} & 5x & - & 4y & = & 1 \\ + & 14x & + & 4y & = & 94 \\ \hline & 19x & & & = & 95 \end{array}$$

and so

$$x = \frac{95}{19} = 5$$

Now that we have a value for x we can substitute this into Equation (2) in order to find y. Substituting

$$\begin{aligned} 5x - 4y &= 1 \\ 5 \times 5 - 4y &= 1 \\ 25 &= 4y + 1 \\ 24 &= 4y \\ y &= 6 \end{aligned}$$

The solution is $x = 5$, $y = 6$.

4. Examples

Solve the simultaneous equations

$$3x + 7y = 27 \quad (1)$$

$$5x + 2y = 16 \quad (2)$$

We will multiply Equation (1) by 5 and Equation (2) by 3 because this will make the coefficients of x in both equations the same.

$$15x + 35y = 135 \quad (3)$$

$$15x + 6y = 48 \quad (4)$$

If we now subtract Equation (4) from Equation (3) we can eliminate the terms involving x.

$$\begin{array}{rcrcr} & 15x & + & 35y & = & 135 \\ - & 15x & + & 6y & = & 48 \\ \hline & & & 29y & = & 87 \end{array}$$

from which

$$y = \frac{87}{29} = 3$$

If we substitute this result in Equation (1) we can find x.

$$\begin{aligned} 3x + 7y &= 27 \\ 3x + 21 &= 27 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

As before, the solution should be checked by substitution into the original equations. So, with $x = 2$, $y = 3$, the left-hand side of Equation (1) is $3(2) + 7(3) = 27$, which is the same as the right-hand side. With $x = 2$, $y = 3$, the left-hand side of Equation (2) is $5(2) + 2(3) = 16$, which is the same as the right-hand side.

All the examples that we have looked at so far have all had whole number coefficients; let's have a look at a couple that don't look like the ones we have just done.

Example

Solve the simultaneous equations

$$\begin{aligned} x &= 3y \\ \frac{x}{3} - y &= 34 \end{aligned}$$

First of all let us rearrange the first equation so that x and y terms are on the left. We will also multiply the second equation by 3 to remove the fraction. These operations give

$$\begin{aligned} x - 3y &= 0 \\ x - 3y &= 102 \end{aligned}$$

Notice that the terms on the left in both equations are exactly the same. If we subtract the equations we will find $0 = -102$. This does not make sense. Remember right at the beginning of this unit we explained that if two lines are parallel they will not intersect. This is the case here. There are no solutions.

Example

$$\frac{x}{5} - \frac{y}{4} = 0 \qquad (1)$$

$$3x + \frac{1}{2}y = 17 \qquad (2)$$

Observe that if both sides of Equation (1) are multiplied by 20 we can remove the fractions:

$$4x - 5y = 0 \qquad (3)$$

If Equation (2) is multiplied by 2 we can remove the fraction there too.

$$6x + y = 34 \qquad (4)$$

Now multiply Equation (4) by 5:

$$30x + 5y = 170 \qquad (5)$$

We can now add (3) and (5) to obtain

$$34x = 170$$
$$x = \frac{170}{34} = 5$$

Substituting this value into Equation (1) gives

$$\begin{aligned} \frac{x}{5} - \frac{y}{4} &= 0 \\ 1 - \frac{y}{4} &= 0 \end{aligned}$$

from which $y = 4$.

So the solution is: $x = 5, y = 4$. As before, this should be checked by substitution into the original equations. So, with $x = 5$, $y = 4$, the left-hand side of Equation (1) is $\frac{5}{5} - \frac{4}{4} = 0$, which is the same as the right-hand side. With $x = 5$, $y = 4$, the left-hand side of Equation (2) is $3(5) + \frac{1}{2}(4) = 17$, which is the same as the right-hand side.

<u>To summarise:</u>
A pair of simultaneous equations represent two straight lines. In effect when we solve them we are looking for the point where the two straight lines intersect. The method of elimination is much better to use than the first method.

Remember the answer you get can always be checked by substituting the pair of values into the original equations.

Exercises

2. Use elimination to solve the following pairs of simultaneous equations.

a) $\begin{aligned} 5x + 3y &= 9 \\ 2x - 3y &= 12 \end{aligned}$ b) $\begin{aligned} 2x - 3y &= 9 \\ 2x + y &= 13 \end{aligned}$ c) $\begin{aligned} x + 7y &= 10 \\ 3x - 2y &= 7 \end{aligned}$

d) $\begin{aligned} 5x + y &= 10 \\ 7x - 3y &= 14 \end{aligned}$ e) $\begin{aligned} \tfrac{1}{3}x + y &= \tfrac{10}{3} \\ 2x + \tfrac{1}{4}y &= \tfrac{11}{4} \end{aligned}$ f) $\begin{aligned} 3x - 2y &= \tfrac{5}{2} \\ \tfrac{1}{3}x + 3y &= -\tfrac{4}{3} \end{aligned}$

3. Solve the following pairs of simultaneous equations by a method of your choice.

a) $\begin{aligned} x &= 3y \\ 4x - 5y &= 35 \end{aligned}$ b) $\begin{aligned} x &= \tfrac{1}{3}y \\ 2y - 6x &= 9 \end{aligned}$ c) $\begin{aligned} 7x + 3y &= -15 \\ 12y - 5x &= 39 \end{aligned}$

Answers

1. a) $x = 2, y = 7$ b) $x = 1, y = 2$ c) $x = 1, y = -2$
   d) $x = 10, y = 3$ e) $x = 1/3, y = 1$ f) $x = 3/4, y = 2/3$

2. a) $x = 3, y = -2$ b) $x = 6, y = 1$ c) $x = 3, y = 1$
   d) $x = 2, y = 0$ e) $x = 1, y = 3$ f) $x = 1/2, y = -1/2$

3. a) $x = 15, y = 5$ b) no solution c) $x = -3, y = 2$

