Simplification and Factorisation

1.3

Introduction

In this Section we explain what is meant by the phrase 'like terms' and show how like terms are collected together and simplified.

Next we consider removing brackets. In order to simplify an expression which contains brackets it is often necessary to rewrite the expression in an equivalent form but without any brackets. This process of removing brackets must be carried out according to particular rules which are described in this Section.

Finally, factorisation, which can be considered as the reverse of the process, is dealt with. It is essential that you have had plenty practice in removing brackets before you study factorisation.

Prerequisites Before starting this Section you should . . .	• be familiar with algebraic notation • have competence in removing brackets
Learning Outcomes On completion you should be able to . . .	• use the laws of indices • simplify expressions by collecting like terms • use the laws of indices • identify common factors in an expression • factorise simple expressions • factorise quadratic expressions

1. Addition and subtraction of like terms

Like terms are multiples of the same quantity. For example $5y$, $17y$ and $\frac{1}{2}y$ are all multiples of y and so are like terms. Similarly, $3x^2$, $-5x^2$ and $\frac{1}{4}x^2$ are all multiples of x^2 and so are like terms.

Further examples of like terms are:

kx and ℓx which are both multiples of x,

x^2y, $6x^2y$, $-13x^2y$, $-2yx^2$, which are all multiples of x^2y

abc^2, $-7abc^2$, $kabc^2$, are all multiples of abc^2

Like terms can be added or subtracted in order to simplify expressions.

Example 27

Simplify $5x - 13x + 22x$.

Solution

All three terms are multiples of x and so are like terms. The expression can be simplified to $14x$.

Example 28

Simplify $5z + 2x$.

Solution

$5z$ and $2x$ are not like terms. They are not multiples of the same quantity. This expression cannot be simplified.

Task

Simplify $5a + 2b - 7a - 9b$.

Your solution

$5a + 2b - 7a - 9b =$

Answer

$-2a - 7b$

Example 29
Simplify $2x^2 - 7x + 11x^2 + x$.

Solution

$2x^2$ and $11x^2$, both being multiples of x^2, can be collected together and added to give $13x^2$.

Similarly, $-7x$ and x can be added to give $-6x$.

We get $2x^2 - 7x + 11x^2 + x = 13x^2 - 6x$ which cannot be simplified further.

Task

Simplify $\frac{1}{2}x + \frac{3}{4}x - 2y$.

Your solution

$\frac{1}{2}x + \frac{3}{4}x - 2y =$

Answer

$\frac{5}{4}x - 2y$

Example 30
Simplify $3a^2b - 7a^2b - 2b^2 + a^2$.

Solution

Note that $3a^2b$ and $7a^2b$ are both multiples of a^2b and so are like terms. There are no other like terms. Therefore

$$3a^2b - 7a^2b - 2b^2 + a^2 = -4a^2b - 2b^2 + a^2$$

Exercises

1. Simplify, if possible,

   (a) $5x + 2x + 3x$, (b) $3q - 2q + 11q$, (c) $7x^2 + 11x^2$, (d) $-11v^2 + 2v^2$, (e) $5p + 3q$

2. Simplify, if possible, (a) $5w + 3r - 2w + r$, (b) $5w^2 + w + 1$, (c) $6w^2 + w^2 - 3w^2$

3. Simplify, if possible,

   (a) $7x + 2 + 3x + 8x - 11$, (b) $2x^2 - 3x + 6x - 2$, (c) $-5x^2 - 3x^2 + 11x + 11$,

   (d) $4q^2 - 4r^2 + 11r + 6q$, (e) $a^2 + ba + ab + b^2$, (f) $3x^2 + 4x + 6x + 8$,

   (g) $s^3 + 3s^2 + 2s^2 + 6s + 4s + 12$.

4. Explain the distinction, if any, between each of the following expressions, and simplify if possible.

   (a) $18x - 9x$, (b) $18x(9x)$, (c) $18x(-9x)$, (d) $-18x - 9x$, (e) $-18x(9x)$

5. Explain the distinction, if any, between each of the following expressions, and simplify if possible.

   (a) $4x - 2x$, (b) $4x(-2x)$, (c) $4x(2x)$, (d) $-4x(2x)$, (e) $-4x - 2x$, (f) $(4x)(2x)$

6. Simplify, if possible,

   (a) $\dfrac{2}{3}x^2 + \dfrac{x^2}{3}$, (b) $0.5x^2 + \frac{3}{4}x^2 - \frac{11}{2}x$, (c) $3x^3 - 11x + 3yx + 11$,

   (d) $-4\alpha x^2 + \beta x^2$ where α and β are constants.

Answers

1. (a) $10x$, (b) $12q$, (c) $18x^2$, (d) $-9v^2$, (e) cannot be simplified.

2. (a) $3w + 4r$, (b) cannot be simplified, (c) $4w^2$

3. (a) $18x - 9$, (b) $2x^2 + 3x - 2$, (c) $-8x^2 + 11x + 11$, (d) cannot be simplified,

   (e) $a^2 + 2ab + b^2$, (f) $3x^2 + 10x + 8$, (g) $s^3 + 5s^2 + 10s + 12$

4. (a) $9x$, (b) $162x^2$, (c) $-162x^2$, (d) $-27x$, (e) $-162x^2$

5. (a) $4x - 2x = 2x$, (b) $4x(-2x) = -8x^2$, (c) $4x(2x) = 8x^2$, (d) $-4x(2x) = -8x^2$,

   (e) $-4x - 2x = -6x$, (f) $(4x)(2x) = 8x^2$

6. (a) x^2, (b) $1.25x^2 - \dfrac{11}{2}x$, (c) cannot be simplified, (d) $(\beta - 4\alpha)x^2$

2. Removing brackets from expressions $a(b+c)$ and $a(b-c)$

Removing brackets means **multiplying out**. For example $5(2+4) = 5\times 2+5\times 4 = 10+20 = 30$. In this simple example we could alternatively get the same result as follows: $5(2+4) = 5\times 6 = 30$. That is:

$$5(2+4) = 5\times 2+5\times 4$$

In an expression such as $5(x+y)$ it is intended that the 5 multiplies both x and y to produce $5x+5y$. Thus the expressions $5(x+y)$ and $5x+5y$ are equivalent. In general we have the following rules known as **distributive laws**:

Key Point 14

$$a(b+c) = ab+ac$$

$$a(b-c) = ab-ac$$

Note that when the brackets are removed both terms in the brackets are multiplied by a.

As we have noted above, if you insert numbers instead of letters into these expressions you will see that both left and right hand sides are equivalent. For example

$4(3+5)$ has the same value as $4(3)+4(5)$, that is 32

and

$7(8-3)$ has the same value as $7(8)-7(3)$, that is 35

Example 31

Remove the brackets from (a) $9(2+y)$, (b) $9(2y)$.

Solution

(a) In the expression $9(2+y)$ the 9 must multiply both terms in the brackets:

$$\begin{aligned} 9(2+y) &= 9(2)+9(y) \\ &= 18+9y \end{aligned}$$

(b) Recall that $9(2y)$ means $9\times(2\times y)$ and that when multiplying numbers together the presence of brackets is irrelevant. Thus $9(2y) = 9\times 2\times y = 18y$

The crucial distinction between the role of the factor 9 in the two expressions $9(2+y)$ and $9(2y)$ in Example 31 should be noted.

Example 32
Remove the brackets from $9(x+2y)$.

Solution

In the expression $9(x+2y)$ the 9 must multiply both the x and the $2y$ in the brackets. Thus

$$\begin{aligned} 9(x+2y) &= 9x+9(2y) \\ &= 9x+18y \end{aligned}$$

Task

Remove the brackets from $9(2x+3y)$.

Remember that the 9 must multiply both the term $2x$ and the term $3y$:

Your solution

$9(2x+3y) =$

Answer

$18x+27y$

Example 33
Remove the brackets from $-3(5x-z)$.

Solution

The number -3 must multiply both the $5x$ and the z.

$$\begin{aligned} -3(5x-z) &= (-3)(5x)-(-3)(z) \\ &= -15x+3z \end{aligned}$$

Task

Remove the brackets from $6x(3x - 2y)$.

Your solution

Answer

$$6x(3x - 2y) = 6x(3x) - 6x(2y) = 18x^2 - 12xy$$

Example 34

Remove the brackets from $-(3x + 1)$.

Solution

Although the 1 is unwritten, the minus sign outside the brackets stands for -1. We must therefore consider the expression $-1(3x + 1)$.

$$\begin{aligned} -1(3x+1) &= (-1)(3x) + (-1)(1) \\ &= -3x + (-1) \\ &= -3x - 1 \end{aligned}$$

Task

Remove the brackets from $-(5x - 3y)$.

Your solution

Answer

$-(5x - 3y)$ means $-1(5x - 3y)$.

$$-1(5x - 3y) = (-1)(5x) - (-1)(3y) = -5x + 3y$$

Task

Remove the brackets from $m(m-n)$.

In the expression $m(m-n)$ the first m must multiply both terms in the brackets:

Your solution

$m(m-n) =$

Answer

$m^2 - mn$

Example 35

Remove the brackets from the expression $5x - (3x+1)$ and simplify the result by collecting like terms.

Solution

The brackets in $-(3x+1)$ were removed in Example 34 on page 46.

$$\begin{aligned} 5x - (3x+1) &= 5x - 1(3x+1) \\ &= 5x - 3x - 1 \\ &= 2x - 1 \end{aligned}$$

Example 36

Show that $\dfrac{-x-1}{4}$, $\dfrac{-(x+1)}{4}$ and $-\dfrac{x+1}{4}$ are all equivalent expressions.

Solution

Consider $-(x+1)$. Removing the brackets we obtain $-x-1$ and so

$$\frac{-x-1}{4} \quad \text{is equivalent to} \quad \frac{-(x+1)}{4}$$

A negative quantity divided by a positive quantity will be negative. Hence

$$\frac{-(x+1)}{4} \quad \text{is equivalent to} \quad -\frac{x+1}{4}$$

You should study all three expressions carefully to recognise the variety of equivalent ways in which we can write an algebraic expression.

Sometimes the bracketed expression can appear on the left, as in $(a+b)c$. To remove the brackets here we use the following rules:

Key Point 15

$$(a+b)c = ac + bc$$

$$(a-b)c = ac - bc$$

Note that when the brackets are removed both the terms in the brackets multiply c.

Example 37

Remove the brackets from $(2x+3y)x$.

Solution

Both terms in the brackets multiply the x outside. Thus

$$\begin{aligned}(2x+3y)x &= 2x(x) + 3y(x)\\ &= 2x^2 + 3yx\end{aligned}$$

Task

Remove the brackets from (a) $(x+3)(-2)$, (b) $(x-3)(-2)$.

Your solution

(a) $(x+3)(-2) =$

Answer

Both terms in the bracket must multiply the -2, giving $-2x-6$

Your solution

(b) $(x-3)(-2) =$

Answer

$-2x+6$

3. Removing brackets from expressions of the form $(a+b)(c+d)$

Sometimes it is necessary to consider two bracketed terms multiplied together. In the expression $(a+b)(c+d)$, by regarding the first bracket as a single term we can use the result in Key Point 14 to write it as $(a+b)c+(a+b)d$. Removing the brackets from each of these terms produces $ac+bc+ad+bd$. More concisely:

Key Point 16

$$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd$$

We see that each term in the first bracketed expression multiplies each term in the second bracketed expression.

Example 38

Remove the brackets from $(3+x)(2+y)$

Solution

We find $$\begin{aligned}(3+x)(2+y) &= (3+x)(2) + (3+x)y \\ &= (3)(2) + (x)(2) + (3)(y) + (x)(y) = 6 + 2x + 3y + xy\end{aligned}$$

Example 39

Remove the brackets from $(3x+4)(x+2)$ and simplify your result.

Solution

$$\begin{aligned}(3x+4)(x+2) &= (3x+4)(x) + (3x+4)(2) \\ &= 3x^2 + 4x + 6x + 8 = 3x^2 + 10x + 8\end{aligned}$$

Example 40

Remove the brackets from $(a+b)^2$ and simplify your result.

Solution

When a quantity is squared it is multiplied by itself. Thus

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= (a+b)a + (a+b)b \\ &= a^2 + ba + ab + b^2 = a^2 + 2ab + b^2\end{aligned}$$

Key Point 17

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Task

Remove the brackets from the following expressions and simplify the results.

(a) $(x+7)(x+3)$, (b) $(x+3)(x-2)$,

Your solution

(a) $(x+7)(x+3) =$

Answer

$x^2 + 7x + 3x + 21 = x^2 + 10x + 21$

Your solution

(b) $(x+3)(x-2) =$

Answer

$x^2 + 3x - 2x - 6 = x^2 + x - 6$

Example 41

Explain the distinction between $(x+3)(x+2)$ and $x+3(x+2)$.

Solution

In the first expression removing the brackets we find

$$\begin{aligned}(x+3)(x+2) &= x^2+3x+2x+6\\ &= x^2+5x+6\end{aligned}$$

In the second expression we have

$$x+3(x+2) = x+3x+6 = 4x+6$$

Note that in the second expression the term $(x+2)$ is only multiplied by 3 and not by x.

Example 42

Remove the brackets from $(s^2+2s+4)(s+3)$.

Solution

Each term in the first bracket must multiply each term in the second. Working through all combinations systematically we have

$$\begin{aligned}(s^2+2s+4)(s+3) &= (s^2+2s+4)(s) \quad + \quad (s^2+2s+4)(3)\\ &= s^3+2s^2+4s+3s^2+6s+12\\ &= s^3+5s^2+10s+12\end{aligned}$$

Engineering Example 1

Reliability in a communication network

Introduction

The reliability of a communication network depends on the reliability of its component parts. The reliability of a component can be represented by a number between 0 and 1 which represents the probability that it will function over a given period of time.

A very simple system with only two components C_1 and C_2 can be configured in series or in parallel. If the components are in **series** then the system will fail if one component fails (see Figure 4)

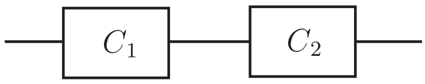

Figure 4: Both components 1 and 2 must function for the system to function

If the components are in **parallel** then only one component need function properly (see Figure 5) and we have built-in redundancy.

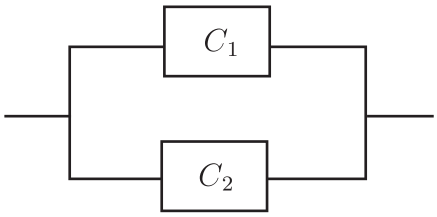

Figure 5: Either component 1 or 2 must function for the system to function

The reliability of a system with two units in parallel is given by $1 - (1 - R_1)(1 - R_2)$ which is the same as $R_1 + R_2 - R_1R_2$, where R_i is the reliability of component C_i. The reliability of a system with 3 units in parallel, as in Figure 6, is given by

$$1 - (1 - R_1)(1 - R_2)(1 - R_3)$$

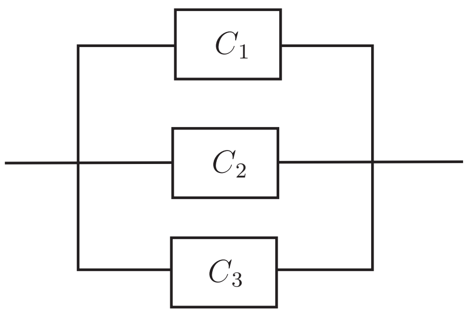

Figure 6: At least one of the three components must function for the system to function

Problem in words

(a) Show that the expression for the system reliability for three components in parallel is equal to $R_1 + R_2 + R_3 - R_1R_2 - R_1R_3 - R_2R_3 + R_1R_2R_3$

(b) Find an expression for the reliability of the system when the reliability of each of the components is the same i.e. $R_1 = R_2 = R_3 = R$

(c) Find the system reliability when $R = 0.75$

(d) Find the system reliability when there are two parallel components each with reliability $R = 0.75$.

Mathematical statement of the problem

(a) Show that $1-(1-R_1)(1-R_2)(1-R_3) \equiv R_1+R_2+R_3-R_1R_2-R_1R_3-R_2R_3+R_1R_2R_3$

(b) Find $1-(1-R_1)(1-R_2)(1-R_3)$ in terms of R when $R_1 = R_2 = R_3 = R$

(c) Find the value of (b) when $R = 0.75$

(d) Find $1-(1-R_1)(1-R_2)$ when $R_1 = R_2 = 0.75$.

Mathematical analysis

(a)
$$\begin{aligned}
1-(1-R_1)(1-R_2)(1-R_3) &\equiv 1-(1-R_1-R_2+R_1R_2)(1-R_3)\\
&= 1-((1-R_1-R_2+R_1R_2)\times 1-(1-R_1-R_2+R_1R_2)\times R_3)\\
&= 1-(1-R_1-R_2+R_1R_2-(R_3-R_1R_3-R_2R_3+R_1R_2R_3))\\
&= 1-(1-R_1-R_2+R_1R_2-R_3+R_1R_3+R_2R_3-R_1R_2R_3)\\
&= 1-1+R_1+R_2-R_1R_2+R_3-R_1R_3-R_2R_3+R_1R_2R_3\\
&= R_1+R_2+R_3-R_1R_2-R_1R_3-R_2R_3+R_1R_2R_3
\end{aligned}$$

(b) When $R_1 = R_2 = R_3 = R$ the reliability is

$1-(1-R)^3$ which is equivalent to $3R-3R^2+R^3$

(c) When $R_1 = R_2 = R_3 = 0.75$ we get

$$1-(1-0.75)^3 = 1-0.25^3 = 1-0.015625 = 0.984375$$

(d) $1-(0.25)^2 = 0.9375$

Interpretation

The mathematical analysis confirms the expectation that the more components there are in parallel then the more reliable the system becomes (1 component: 0.75; 2 components: 0.9375; 3 components: 0.984375). With three components in parallel, as in part (c), although each individual component is relatively unreliable ($R = 0.75$ implies a one in four chance of failure of an individual component) the system as a whole has an over 98% probability of functioning (under 1 in 50 chance of failure).

Exercises

1. Remove the brackets from each of the following expressions:

   (a) $2(mn)$, (b) $2(m+n)$, (c) $a(mn)$, (d) $a(m+n)$, (e) $a(m-n)$,
   (f) $(am)n$, (g) $(a+m)n$, (h) $(a-m)n$, (i) $5(pq)$, (j) $5(p+q)$,
   (k) $5(p-q)$, (l) $7(xy)$, (m) $7(x+y)$, (n) $7(x-y)$, (o) $8(2p+q)$,
   (p) $8(2pq)$, (q) $8(2p-q)$, (r) $5(p-3q)$, (s) $5(p+3q)$ (t) $5(3pq)$.

2. Remove the brackets from each of the following expressions:

   (a) $4(a+b)$, (b) $2(m-n)$, (c) $9(x-y)$,

3. Remove the brackets from each of the following expressions and simplify where possible:

   (a) $(2+a)(3+b)$, (b) $(x+1)(x+2)$, (c) $(x+3)(x+3)$, (d) $(x+5)(x-3)$

4. Remove the brackets from each of the following expressions:

   (a) $(7+x)(2+x)$, (b) $(9+x)(2+x)$, (c) $(x+9)(x-2)$, (d) $(x+11)(x-7)$,
   (e) $(x+2)x$, (f) $(3x+1)x$, (g) $(3x+1)(x+1)$, (h) $(3x+1)(2x+1)$,
   (i) $(3x+5)(2x+7)$, (j) $(3x+5)(2x-1)$, (k) $(5-3x)(x+1)$ (l) $(2-x)(1-x)$.

5. Remove the brackets from $(s+1)(s+5)(s-3)$.

Answers

1. (a) $2mn$, (b) $2m+2n$, (c) amn, (d) $am+an$, (e) $am-an$, (f) amn, (g) $an+mn$,
   (h) $an-mn$, (i) $5pq$, (j) $5p+5q$, (k) $5p-5q$, (l) $7xy$, (m) $7x+7y$, (n) $7x-7y$,
   (o) $16p+8q$, (p) $16pq$, (q) $16p-8q$, (r) $5p-15q$, (s) $5p+15q$, (t) $15pq$

2. (a) $4a+4b$, (b) $2m-2n$, (c) $9x-9y$

3. (a) $6+3a+2b+ab$, (b) x^2+3x+2, (c) x^2+6x+9, (d) $x^2+2x-15$

4. On removing brackets we obtain:

   (a) $14+9x+x^2$, (b) $18+11x+x^2$, (c) $x^2+7x-18$, (d) $x^2+4x-77$
   (e) x^2+2x, (f) $3x^2+x$, (g) $3x^2+4x+1$ (h) $6x^2+5x+1$
   (i) $6x^2+31x+35$, (j) $6x^2+7x-5$, (k) $-3x^2+2x+5$, (l) x^2-3x+2

5. $s^3+3s^2-13s-15$

4. Factorisation

A number is said to be **factorised** when it is written as a product. For example, 21 can be factorised into 7×3. We say that 7 and 3 are **factors** of 21.

Algebraic expressions can also be factorised. Consider the expression $7(2x + 1)$. Removing the brackets we can rewrite this as

$$7(2x + 1) = 7(2x) + (7)(1) = 14x + 7.$$

Thus $14x + 7$ is equivalent to $7(2x + 1)$. We see that $14x + 7$ has factors 7 and $(2x + 1)$. The factors 7 and $(2x + 1)$ *multiply* together to give $14x + 7$. The process of writing an expression as a product of its factors is called **factorisation**. When asked to factorise $14x + 7$ we write

$$14x + 7 = 7(2x + 1)$$

and so we see that factorisation can be regarded as reversing the process of removing brackets.

Always remember that the factors of an algebraic expression are *multiplied* together.

Example 43

Factorise the expression $4x + 20$.

Solution

Both terms in the expression $4x + 20$ are examined to see if they have any factors in common. Clearly 20 can be factorised as $(4)(5)$ and so we can write

$$4x + 20 = 4x + (4)(5)$$

The factor 4 is common to both terms on the right; it is called a **common factor** and is placed at the front and outside brackets to give

$$4x + 20 = 4(x + 5)$$

Note that the factorised form can be checked by removing the brackets again.

Example 44

Factorise $z^2 - 5z$.

Solution

Note that since $z^2 = z \times z$ we can write

$$z^2 - 5z = z(z) - 5z$$

so that there is a common factor of z. Hence

$$z^2 - 5z = z(z) - 5z = z(z - 5)$$

Example 45

Factorise $6x - 9y$.

Solution

By observation, we see that there is a common factor of 3. Thus $6x - 9y = 3(2x - 3y)$

Task

Factorise $14z + 21w$.

(a) Find the factor common to both $14z$ and $21w$:

Your solution

Answer

7

(b) Now factorise $14z + 21w$:

Your solution

$14z + 21w =$

Answer

$7(2z + 3w)$

Note: If you have any doubt, you can check your answer by removing the brackets again.

Task

Factorise $6x - 12xy$.

First identify the two common factors:

Your solution

Answer

6 and x

Now factorise $6x - 12xy$:

Your solution

$6x - 12xy =$

Answer

$6x(1 - 2y)$

Exercises

1. Factorise

   (a) $5x + 15y$, (b) $3x - 9y$, (c) $2x + 12y$, (d) $4x + 32z + 16y$, (e) $\frac{1}{2}x + \frac{1}{4}y$.

   In each case check your answer by removing the brackets again.

2. Factorise

   (a) $a^2 + 3ab$, (b) $xy + xyz$, (c) $9x^2 - 12x$

3. Explain why a is a factor of $a + ab$ but b is not. Factorise $a + ab$.

4. Explain why x^2 is a factor of $4x^2 + 3yx^3 + 5yx^4$ but y is not.

   Factorise $4x^2 + 3yx^3 + 5yx^4$.

Answers

1. (a) $5(x + 3y)$, (b) $3(x - 3y)$, (c) $2(x + 6y)$, (d) $4(x + 8z + 4y)$, (e) $\frac{1}{2}(x + \frac{1}{2}y)$
2. (a) $a(a + 3b)$, (b) $xy(1 + z)$, (c) $3x(3x - 4)$.
3. $a(1 + b)$.
4. $x^2(4 + 3yx + 5yx^2)$.

5. Factorising quadratic expressions

Quadratic expressions commonly occur in many areas of mathematics, physics and engineering. Many quadratic expressions can be written as the product of two linear factors and, in this Section, we examine how these factors can be easily found.

Key Point 18

An expression of the form

$$ax^2 + bx + c \qquad a \neq 0$$

where a, b and c are numbers is called a **quadratic expression** (in the variable x).

The numbers b and c may be zero but a must not be zero (for, then, the quadratic reduces to a linear expression or constant). The number a is called the **coefficient** of x^2, b is the coefficient of x and c is called the **constant term**.

Case 1

Consider the product $(x+1)(x+2)$. Removing brackets yields x^2+3x+2. Conversely, we see that the factors of x^2+3x+2 are $(x+1)$ and $(x+2)$. However, if we were given the quadratic expression first, how would we factorise it ? The following examples show how to do this but note that not all quadratic expressions can be easily factorised.

To enable us to factorise a quadratic expression in which the coefficient of x^2 equals 1, we note the following expansion:

$$(x+m)(x+n) = x^2 + mx + nx + mn = x^2 + (m+n)x + mn$$

So, given a quadratic expression we can think of the coefficient of x as $m+n$ and the constant term as mn. Once the values of m and n have been found the factors can be easily obtained.

Example 46

Factorise $x^2 + 4x - 5$.

Solution

Writing $x^2 + 4x - 5 = (x+m)(x+n) = x^2 + (m+n)x + mn$ we seek numbers m and n such that $m+n = 4$ and $mn = -5$. By trial and error it is not difficult to find that $m = 5$ and $n = -1$ (or, the other way round, $m = -1$ and $n = 5$). So we can write

$$x^2 + 4x - 5 = (x+5)(x-1)$$

The answer can be checked easily by removing brackets.

Task

Factorise $x^2 + 6x + 8$.

As the coefficient of x^2 is 1, we can write

$$x^2 + 6x + 8 = (x+m)(x+n) = x^2 + (m+n)x + mn$$

so that $m+n = 6$ and $mn = 8$.

First, find suitable values for m and n:

Your solution

Answer

$m = 4,\ n = 2$ or, the other way round, $m = 2,\ n = 4$

Finally factorise the quadratic:

Your solution

$x^2 + 6x + 8 =$

Answer

$(x + 4)(x + 2)$

Case 2

When the coefficient of x^2 is not equal to 1 it may be possible to extract a *numerical* factor. For example, note that $3x^2 + 18x + 24$ can be written as $3(x^2 + 6x + 8)$ and then factorised as in the previous Task in Case 1. Sometimes no numerical factor can be found and a slightly different approach may be taken. We will demonstrate a technique which can always be used to transform the given expression into one in which the coefficient of the squared variable equals 1.

Example 47

Factorise $2x^2 + 5x + 3$.

Solution

First note the coefficient of x^2; in this case 2. Multiply the whole expression by this number and rearrange as follows:

$$2(2x^2 + 5x + 3) = 2(2x^2) + 2(5x) + 2(3) = (2x)^2 + 5(2x) + 6.$$

We now introduce a new variable z such that $z = 2x$ Thus we can write

$$(2x)^2 + 5(2x) + 6 \qquad \text{as} \qquad z^2 + 5z + 6$$

This can be factorised to give $(z + 3)(z + 2)$. Returning to the original variable by replacing z by $2x$ we find

$$2(2x^2 + 5x + 3) = (2x + 3)(2x + 2)$$

A factor of 2 can be extracted from the second bracket on the right so that

$$2(2x^2 + 5x + 3) = 2(2x + 3)(x + 1)$$

so that

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

As an alternative to the technique of Example 47, experience and practice can often help us to identify factors. For example suppose we wish to factorise $3x^2 + 7x + 2$. We write

$$3x^2 + 7x + 2 = (\qquad\qquad)(\qquad\qquad)$$

In order to obtain the term $3x^2$ we can place terms $3x$ and x in the brackets to give

$$3x^2 + 7x + 2 = (3x + \quad ? \quad)(x + \quad ? \quad)$$

In order to obtain the constant 2, we consider the factors of 2. These are 1,2 or -1,-2. By placing these factors in the brackets we can factorise the quadratic expression. Various possibilities exist: we could write $(3x+2)(x+1)$ or $(3x+1)(x+2)$ or $(3x-2)(x-1)$ or $(3x-1)(x-2)$, only one of which is correct. By removing brackets from each in turn we look for the factorisation which produces the correct middle term, $7x$. The correct factorisation is found to be

$$3x^2 + 7x + 2 = (3x + 1)(x + 2)$$

With practice you will be able to carry out this process quite easily.

Task

Factorise the quadratic expression $5x^2 - 7x - 6$.

Write $5x^2 - 7x - 6 = (\qquad)(\qquad)$

To obtain the quadratic term $5x^2$, insert $5x$ and x in the brackets:

$$5x^2 - 7x - 6 = (5x + \ ?\ )(x + \ ?\ )$$

Now find the factors of -6:

Your solution

Answer

$3, -2$ or $-3, 2$ or $-6, 1$ or $6, -1$

Use these factors in turn to find which pair, if any, gives rise to the middle term, $-7x$, and complete the factorisation:

Your solution

$5x^2 - 7x - 6 = (5x + \ )(x + \ ) =$

Answer

$(5x + 3)(x - 2)$

On occasions you will meet expressions of the form $x^2 - y^2$ known as the **difference of two squares**. It is easy to verify by removing brackets that this factorises as

$$x^2 - y^2 = (x + y)(x - y)$$

So, if you can learn to recognise such expressions it is an easy matter to factorise them.

Example 48

Factorise

(a) $x^2 - 36z^2$, (b) $25x^2 - 9z^2$, (c) $\alpha^2 - 1$

Solution

In each case we are required to find the difference of two squared terms.

(a) Note that $x^2 - 36z^2 = x^2 - (6z)^2$. This factorises as $(x + 6z)(x - 6z)$.

(b) Here $25x^2 - 9z^2 = (5x)^2 - (3z)^2$. This factorises as $(5x + 3z)(5x - 3z)$.

(c) $\alpha^2 - 1 = (\alpha + 1)(\alpha - 1)$.

Exercises

1. Factorise

   (a) $x^2 + 8x + 7$, (b) $x^2 + 6x - 7$, (c) $x^2 + 7x + 10$, (d) $x^2 - 6x + 9$.

2. Factorise

   (a) $2x^2 + 3x + 1$, (b) $2x^2 + 4x + 2$, (c) $3x^2 - 3x - 6$, (d) $5x^2 - 4x - 1$, (e) $16x^2 - 1$,

   (f) $-x^2 + 1$, (g) $-2x^2 + x + 3$.

3. Factorise

   (a) $x^2 + 9x + 14$, (b) $x^2 + 11x + 18$, (c) $x^2 + 7x - 18$, (d) $x^2 + 4x - 77$,
   (e) $x^2 + 2x$, (f) $3x^2 + x$, (g) $3x^2 + 4x + 1$, (h) $6x^2 + 5x + 1$,
   (i) $6x^2 + 31x + 35$, (j) $6x^2 + 7x - 5$, (k) $-3x^2 + 2x + 5$, (l) $x^2 - 3x + 2$.

4. Factorise (a) $z^2 - 144$, (b) $z^2 - \frac{1}{4}$, (c) $s^2 - \frac{1}{9}$

Answers

1. (a) $(x + 7)(x + 1)$, (b) $(x + 7)(x - 1)$, (c) $(x + 2)(x + 5)$, (d) $(x - 3)(x - 3)$

2. (a) $(2x + 1)(x + 1)$, (b) $2(x + 1)^2$, (c) $3(x + 1)(x - 2)$, (d)$(5x + 1)(x - 1)$,

   (e) $(4x + 1)(4x - 1)$, (f) $(x + 1)(1 - x)$, (g) $(x + 1)(3 - 2x)$

3. The factors are:

   (a) $(7 + x)(2 + x)$, (b) $(9 + x)(2 + x)$, (c) $(x + 9)(x - 2)$, (d) $(x + 11)(x - 7)$,
   (e) $(x + 2)x$, (f) $(3x + 1)x$, (g) $(3x + 1)(x + 1)$, (h) $(3x + 1)(2x + 1)$,
   (i) $(3x + 5)(2x + 7)$, (j) $(3x + 5)(2x - 1)$, (k) $(5 - 3x)(x + 1)$, (l) $(2 - x)(1 - x)$.

4. (a) $(z + 12)(z - 12)$, (b) $(z + \frac{1}{2})(z - \frac{1}{2})$, (c) $(s + \frac{1}{3})(s - \frac{1}{3})$