Introduction to Limits

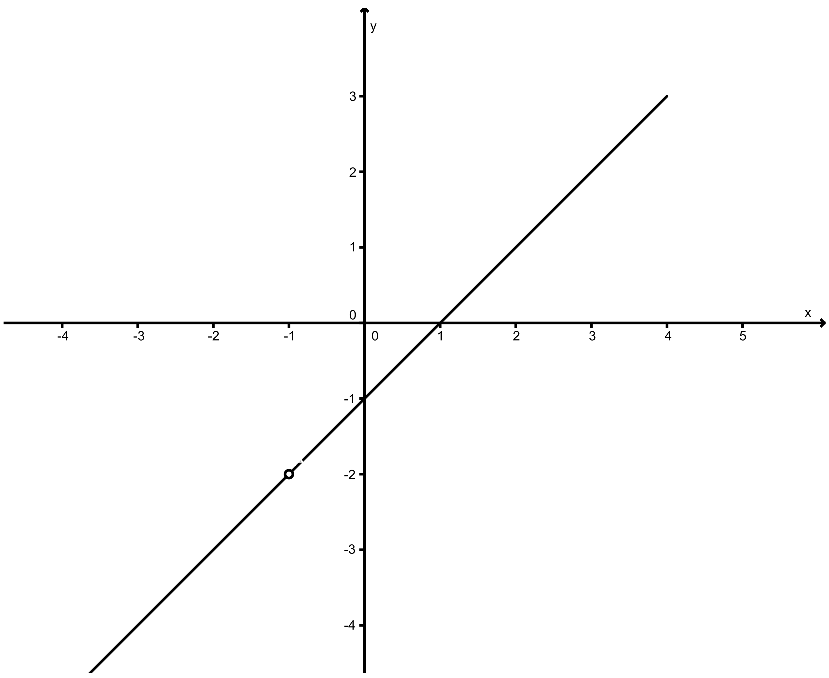
Limits are an extremely useful tool which are used in a wide range of areas in mathematics. For example limits are essential in determining the continuity of functions, sketching graphs, determining the convergence and divergence of sequences or series. Understanding limits can give you an insight into these and many other areas of mathematics.

We write $\lim_{x\to a} f(x) = L$ for a and $L \in \mathbb{R}$, if we can make the values of $f(x)$ as close as we like to L by taking x sufficiently close to a.

Consider the function

$$f(x) = \frac{x^2 - 1}{x + 1}$$

and its graph below.



Material developed by the Department of Mathematics & Statistics, N.U.I. Maynooth and supported by the NDLR (www.ndlr.com).

Notice that $f(-1)$ is undefined, but when $x \neq -1$ we can say that

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x + 1} \\ &= \frac{(x + 1)(x - 1)}{x + 1} \\ &= x - 1. \end{aligned}$$

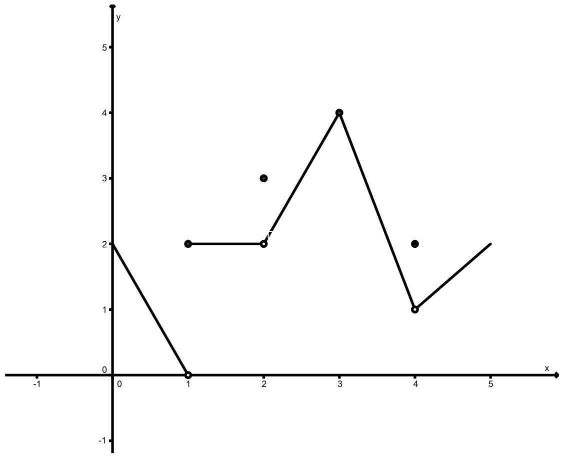
By taking values of x very close to -1, we can make $f(x)$ as close as we like to -2. In this case we write that

$$\lim_{x \to -1} f(x) = -2.$$

Limits of this form will be very important when calculating derivatives.

Difference Between Function Value and Limit

In this handout we ask if $\lim_{x\to a} f(x)$ is always equal to $f(a)$? To answer this question, consider the graph of the function $f(x)$ below.



We will examine the graph in stages.
When x=1:

What is $f(1)$? In order to do this we need to go to an x value of 1 along the x axis and go vertically until you hit the graph. Then go across to find the corresponding y value. If we do this in the graph above we see that

$$f(1) = 2$$

Now suppose we need to find $\lim_{x\to 1^-} f(x)$. In other words, if we were to approach an x value of 1 from the left (numbers smaller than 1), what would the y value approach? Examining the graph we see that

$$\lim_{x\to 1^-} f(x) = 0$$

Next suppose we need to find $\lim_{x\to 1^+} f(x)$. In other words, if we were to approach an x value of 1 from the right (through numbers bigger than 1), what would the y value approach? Once again, examining the graph we see that

$$\lim_{x\to 1^+} f(x) = 2$$

Material developed by the Department of Mathematics & Statistics, N.U.I. Maynooth and supported by the NDLR (www.ndlr.com).

In situations like this, where the limit from the left does not equal the limit from the right, we say that

$$\lim_{x \to 1} f(x) \text{ does not exist.}$$

Note here that $f(1) = 2$ but $\lim_{x \to 1} f(x)$ does not exist.

When x=2:

Examining the graph, and using the same methods as above we can say that:

$$\begin{aligned} f(2) &= 3 \\ \lim_{x \to 2^-} f(x) &= 2 \\ \lim_{x \to 2^+} f(x) &= 2 \end{aligned}$$

In this situation we see that the limit from the left equals the limit from the right. We can therefore say

$$\lim_{x \to 2} f(x) = 2$$

Notice that $f(2) \neq \lim_{x \to 2} f(x)$.

When x=3:

Examining the graph we see:

$$\begin{aligned} f(3) &= 4 \\ \lim_{x \to 3^-} f(x) &= 4 \\ \lim_{x \to 3^+} f(x) &= 4 \end{aligned}$$

In this situation we see that the limit from the left equals the limit from the right. We can therefore say

$$\lim_{x \to 3} f(x) = 4$$

Notice that $f(3) = \lim_{x \to 3} f(x)$. In situations like this, where $\lim_{x \to a} f(x) = f(a)$, we say that $f(x)$ is continuous at $x = a$.

For practice, using the graph above, find:
(a)$f(4)$ (b)$\lim_{x \to 4^-} f(x)$ (c)$\lim_{x \to 4^+} f(x)$ (d) $\lim_{x \to 4} f(x)$

Solutions
(a) 2 (b) 1 (c) 1 (d) 1